

FIELD THEORY AND THE COHOMOLOGY OF SOME GALOIS GROUPS

ALEJANDRO ADEM, WENFENG GAO, DIKRAN B. KARAGUEUZIAN, AND JÁN MINÁČ

ABSTRACT. We prove that two arithmetically significant extensions of a field F coincide if and only if the Witt ring WF is a group ring $\mathbb{Z}/n[G]$. Furthermore, working modulo squares with Galois groups which are 2-groups, we establish a theorem analogous to Hilbert's theorem 90 and show that an identity linking the cohomological dimension of the Galois group of the quadratic closure of F , the length of a filtration on a certain module over a Galois group, and the dimension over \mathbb{F}_2 of the square class group of the field holds for a number of interesting families of fields. Finally, we discuss the cohomology of a particular Galois group in a topological context.

1. INTRODUCTION

In recent years substantial breakthroughs have been made in the study of the cohomology of absolute Galois groups, culminating in Voevodsky's proof of the Milnor conjecture [31]. These results severely restrict possible absolute Galois groups of fields and it seems virtually certain that they will have purely field-theoretic consequences. Such results are not easily derived, however, and in fact only a few theorems of this type have appeared (for some examples the reader may consult [18, 24, 30]). The goal of this paper is to obtain some results in field theory as consequences of Merkurjev's theorem [23]. The techniques we use are somewhat varied; for example, we study the square class group of a field as a module for a Galois group and relate its socle series to the $E_{\infty}^{1,1}$ -term of a spectral sequence. We are then able to obtain information on this term of the spectral sequence using techniques from the theory of binary quadratic forms. In the study of particular examples our techniques become more specialized, for example, at one point we make essential use of the main theorem of local class field theory.

Throughout the paper, F will denote a field whose characteristic is not 2. We write $F^{(2)}$ for the field obtained by adjoining all the square roots of elements of F . Now let $F^{\{3\}} = (F^{(2)})^{(2)}$ and $\text{Gal}(F^{(2)}/F) = G_F^{[2]}$. We introduce the following definition: the *V-group* of F is the Galois group $G_F^{\{3\}} = \text{Gal}(F^{\{3\}}/F)$. In [24] Mináč and Spira defined an extension of F whose Galois group is closely related to the Witt ring of F . Let $F^{(3)}$ be the extension of $F^{(2)}$ obtained by adjoining the square roots of elements $\alpha \in F^{(2)}$ such that $F^{(2)}(\sqrt{\alpha})$ is Galois over F . $F^{(3)}$ is a Galois extension of F and we recall that the *W-group* of F is the Galois group $\mathcal{G}_F = \text{Gal}(F^{(3)}/F)$. Of

The first author was partially supported by an NSF grant, the MPIM-Bonn, and the CRM-Barcelona.

The third author was partially supported by an NSF postdoctoral fellowship, CRM-Barcelona, and MPIM-Bonn.

The fourth author was partially supported by the NSERC and the special Dean of Science fund at UWO. .

course $F^{(3)} \subset F^{\{3\}}$, so that \mathcal{G}_F is a quotient of $G_F^{\{3\}}$. Furthermore, it follows immediately from the definitions that $\mathcal{G}_F = G_F^{\{3\}}$ if and only if $F^{\{3\}} = F^{(3)}$.

In this paper we use tools from the cohomology of groups to study these Galois groups and the field-theoretic information they may contain. Our first result is a characterization of C -fields (i.e. fields F such that the Witt ring WF is isomorphic to a group ring of the form $\mathbb{Z}/n\mathbb{Z}[G]$):

Theorem. (3.1) *The W -group of F is the same as the V -group of F if and only if F is a C -field.*

The proof makes use of a theorem due to Merkurjev as well as a description of the $E_\infty^{1,1}$ term of the mod 2 Lyndon-Hochschild-Serre spectral sequence associated to the Frattini extension for \mathcal{G}_F . We introduce a length invariant associated to F as follows. Let $J := \dot{F}^{(2)}/(\dot{F}^{(2)})^2$, regarded as a $G_F^{[2]}$ -module. The socle series of J ,

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J$$

has finite length l (i.e. $J_l = J$) if \dot{F}/\dot{F}^2 is finite. This length is an invariant of the field, which we will write as $l(F)$.

One of the key results used in our proof is

Proposition. (3.10) *In the mod 2 Lyndon-Hochschild-Serre spectral sequence for the group extension $1 \rightarrow \Phi \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1$, we have $E_\infty^{1,1} \approx J_2/J_1$.*

Next we obtain a result reminiscent of Hilbert's Theorem 90, describing the elements in the kernel of the differential d_2 . While the proof of this theorem depends on Proposition 3.10 and hence indirectly on Merkurjev's theorem, we provide a combinatorial, group-theoretic proof of "half" of the theorem in an appendix (§7).

Let $\{[a_i] \mid i \in I\}$ be a basis for \dot{F}/\dot{F}^2 , which we identify with $H^1(G_F^{[2]}, \mathbb{F}_2)$. Let $\{\sigma_i \mid i \in I\}$ be a minimal set of generators of $G_F^{[2]}$ such that $\sigma_i(\sqrt{a_j})/\sqrt{a_j} = (-1)^{\delta_{i,j}}$, where $i, j \in I$ and $\delta_{i,j}$ is the Kronecker delta function. One may view the a_i as a dual basis to the σ_i .

Theorem. (4.1) *If $\{[j_{i_k}] \mid 1 \leq k \leq l\}$ is a finite set of elements in J_1 , then*

$$\sum_{k=0}^l [a_{i_k}] d_2^{0,1}([j_{i_k}]) = 0$$

if and only if there exists an element $[j] \in J_2$ such that

1. $[\sigma_{i_k}(j)/j] = [j_{i_k}]$ for each $k \in \{1, \dots, l\}$ and
2. $[\sigma_u(j)] = [j]$ for each $u \notin \{i_1, \dots, i_l\}$.

One should observe that this result is a practical criterion for the existence of elements $[j]$ from J_2 with prescribed images under the action of the Galois group $G_F^{[2]}$. We illustrate this in a detailed way in a special example when the W -group is the universal W -group on two generators (see example 4.2). Moreover this theorem fully describes the $G_F^{[2]}$ -module J_2 .

Let $G_q := \text{Gal}(F_q/F)$, where F_q is a quadratic closure of F . Our next result provides a formula for the cohomological dimension of G_q for certain non-formally real fields in terms of the length function $l(F)$:

Theorem. (5.2, 5.3, 5.15) *Let F be a field of which is not formally real, and assume that $|\dot{F}/\dot{F}^2| = 2^n$. Then if F is a C -field, a local field, or a field such that G_q is free, we have $l(F) + \text{cd}(G_q) = n + 1$.*

Finally in our last section we make an explicit analysis of the mod 2 cohomology of the V -group arising when G_q is the free pro-2-group on 2 generators (denoted $V(2)$, a group of order 2^7). We obtain its Poincaré series as well as some rather interesting topological information.

Proposition. (6.4) *The Poincaré series for $H^*(V(2), \mathbb{F}_2)$ is*

$$\frac{1 - x + x^2}{(1 - x)^3(1 - x^2)^2},$$

which expands to $1 + 2x + 6x^2 + 11x^3 + 22x^4 + 36x^5 + 60x^6 + 90x^7 + 135x^8 + 190x^9 + 266x^{10} + \dots$.

Our methods combine techniques from field theory with methods from the cohomology of groups. For background information on Galois cohomology, group cohomology, quadratic form theory, and local class field theory, we refer the reader to [28], [3, 7], [21, 22], and [5, 14] respectively. We have endeavored to provide specific references throughout the paper as often as possible.

This paper is organized as follows: in §2 we describe preliminary definitions and basic facts about the Galois groups considered here; in §3 we prove the characterization of C -fields and introduce the length of F ; in §4 we prove our analogue of Hilbert's Theorem 90; in §5 we discuss our results relating $l(F)$ with $\text{cd}(G_q)$ for examples such as local fields; and finally in §6 we provide a fairly complete cohomological analysis of the group $V(2)$ and of a 5-dimensional crystallographic group associated to it.

A few comments about notation may be helpful. We shall have several uses for the cyclic group of order two, and thus as many notations for it. Generally, we denote this group by $\mathbb{Z}/2\mathbb{Z}$, but when we regard it as the group of square roots of unity in a field, we write it μ_2 , and when we consider the field of two elements, used as coefficients for cohomology, we write \mathbb{F}_2 . Generally, we do not specify the coefficients for our cohomology theories, and in such cases \mathbb{F}_2 -coefficients are to be understood. When we use other coefficient rings, they will be specified.

2. PRELIMINARIES

Let F denote a field of characteristic different from two, and denote by $F^{(2)}$ the field obtained by adjoining all the square roots of elements of F . Now let $F^{\{3\}} = (F^{(2)})^{(2)}$. From Kummer theory it is easy to see that $F^{(2)}$ is a Galois extension of F , and if B is a basis of \dot{F}/\dot{F}^2 , then the Galois group of $F^{(2)}$ over F is $\prod_B \mathbb{Z}/2\mathbb{Z}$. We denote this Galois group by $G_F^{[2]}$. A slightly less obvious fact is

Lemma 2.1. $F^{\{3\}}$ is a Galois extension of F .

Proof. Let $\sigma: F^{\{3\}} \rightarrow \bar{F}$ be an embedding of $F^{\{3\}}$ into an algebraic closure \bar{F} of F containing $F^{\{3\}}$. It is enough to show that for each $\gamma \in F^{(2)}$, $\sigma(\sqrt{\gamma}) \in F^{\{3\}}$. However, $\sigma(\sqrt{\gamma})^2 = \sigma(\gamma)$ and therefore $\sigma(\sqrt{\gamma}) = \pm\sigma(\sqrt{\gamma})$. But $\sigma(\gamma) \in F^{(2)}$, since $F^{(2)}/F$ is Galois, so the lemma follows. \square

Using this we introduce

Definition 2.2. The V -group of F is the Galois group $G_F^{\{3\}} = \text{Gal}(F^{\{3\}}/F)$.

In [24] Mináč and Spira defined the “Witt closure” of F , an extension of F whose Galois group is closely related to the Witt ring of F . Let $F^{(3)}$ be the extension of $F^{(2)}$ obtained by adjoining the square roots of elements $\alpha \in F^{(2)}$ such that $F^{(2)}(\sqrt{\alpha})$ is Galois over F . $F^{(3)}$ is a Galois extension of F and we recall

Definition 2.3. The W -group of F is the Galois group $\mathcal{G}_F = \text{Gal}(F^{(3)}/F)$.

Of course $F^{(3)} \subset F^{\{3\}}$, so that \mathcal{G}_F is a quotient of $G_F^{\{3\}}$. We will frequently make use of this fact without explicit mention. Furthermore, it follows immediately from the definitions that $\mathcal{G}_F = G_F^{\{3\}}$ if and only if $F^{\{3\}} = F^{(3)}$.

Now we introduce the Galois group of the quadratic closure and one of its subgroups:

Definition 2.4. Let $G_q := \text{Gal}(F_q/F)$, where F_q is a quadratic closure of F , and set $G^{(2)} := \text{Gal}(F_q/F^{(2)})$.

Note that $G_q/G^{(2)} \cong G_F^{[2]}$ and that the V -group is determined by G_q , namely if $\Phi(G^{(2)})$ denotes the Frattini subgroup of $G^{(2)}$, then from our definition it follows that

$$G_F^{\{3\}} = G_q/\Phi(G^{(2)}).$$

From the above we obtain that there is an extension of elementary abelian groups

$$1 \rightarrow G_{F^{(2)}}^{[2]} \rightarrow G_F^{\{3\}} \rightarrow G_F^{[2]} \rightarrow 1,$$

where the action on the kernel can be highly non-trivial. We will often abbreviate $G_{F^{(2)}}^{[2]}$ by simply using the symbol A . From the point of view of group theory, the V -group of the field F is simply the extension of $G_F^{[2]}$ obtained by taking the quotient of G_q by the Frattini subgroup of $G^{(2)}$.

We will be especially interested in the case when G_q is a free pro-2-group on n generators. From the analysis in [24] we know that in this case the W -group maps onto any W -group arising from a field with n -dimensional group of square classes, hence it is referred to as the universal W -group $W(n)$. Similarly the V -group arising from this situation will enjoy an analogous universal property and we denote it by $V(n)$.

In this case the group $G^{(2)}$ can be identified with the Frattini subgroup of the free pro-2-group G_q , hence it is a free pro-2-group of rank $2^n(n-1)+1$, where $G_F^{[2]} \cong (\mathbb{Z}/2\mathbb{Z})^n$. More explicitly we have $G^{(2)} = G_q^2[G_q, G_q]$ (in fact, $[G_q, G_q] \subset G_q^2$, so $G^{(2)} = G_q^2$). Our next step is to

“abelianize” this extension, namely we factor out the Frattini subgroup of $G^{(2)}$. This yields the quotient $V(n)$ as an extension

$$1 \rightarrow A \rightarrow V(n) \rightarrow G_F^{[2]} \rightarrow 1.$$

Moreover we can identify the $G_F^{[2]}$ -module A explicitly using the methods in [20]; indeed $A \cong \Omega^2(\mathbb{F}_2)$, the second dimension shift of the trivial module (for the definition of the “Heller translate” Ω^2 see [7, p. 8, v. I]). Alternatively, via a detailed study of higher commutators, one can determine the first two stages in a minimal resolution of the dual of A , which also establishes the isomorphism $A \cong \Omega^2(\mathbb{F}_2)$. The group $V(n)$ corresponds to the unique non-trivial element in $H^2(G_F^{[2]}, \Omega^2(\mathbb{F}_2))$; this element in fact restricts non-trivially on every cyclic subgroup, whence the extension above is totally non-split.

Now for the W -group we have a extension of elementary abelian groups

$$1 \rightarrow \Phi(\mathcal{G}_F) \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1,$$

where Φ denotes the Frattini Subgroup of \mathcal{G}_F . Moreover, elaborating on what we have mentioned this can be expressed as an extension

$$1 \rightarrow U \rightarrow G_F^{\{3\}} \rightarrow \mathcal{G}_F \rightarrow 1$$

where $U = \text{Gal}(F^{\{3\}}/F^{(3)})$.

Definition 2.5. A field whose Witt ring is isomorphic to some group ring of the form $\mathbb{Z}/n\mathbb{Z}[G]$ is said to be a C -field (here n is permitted to take the value 0).

In [32, 1.9] a number of equivalent conditions are given for a field to be a C -field. We remark that the p -adic field \mathbb{Q}_p , where p is an odd prime, is a C -field. Furthermore, $\mathbb{R}((t_1)) \cdots ((t_n))$, $\mathbb{C}((t_1)) \cdots ((t_n))$, and direct limits of such fields, are also C -fields. Further examples can be found in [22, p. 46].

Now suppose that $K \subset L$ is a Galois extension of fields, and that there is a subgroup N of \dot{K}/\dot{K}^2 such that $L = K[\sqrt{N}]$. Again from Kummer theory we know that there is a “perfect pairing”

$$N \times \text{Gal}(L/K) \rightarrow \mu_2.$$

Suppose further that $F \subset K$ is another field, and that the extensions $F \subset K$ and $F \subset L$ are also Galois. Then there is a conjugation action of $\text{Gal}(K/F)$ on $\text{Gal}(L/K)$, and an action of $\text{Gal}(K/F)$ on N . These actions are compatible in the sense that the following lemma holds, see for example [18, p. 101] or [34].

Lemma 2.6. *If $\sigma \in \text{Gal}(K/F)$, $n \in N$, $\gamma \in \text{Gal}(L/K)$, and $\langle \cdot, \cdot \rangle$ is the Kummer pairing, then*

$$\langle \sigma n, \gamma \rangle = \langle n, \gamma^{\sigma^{-1}} \rangle,$$

i.e. the action of $\text{Gal}(K/F)$ on $\text{Gal}(L/K)$ and N is compatible with the Kummer pairing.

3. THE V -GROUP OF A C -FIELD

In this section we will prove one of our main theorems, namely

Theorem 3.1. *If F is a field of characteristic different from two, then the W -group of F is the same as the V -group of F if and only if F is a C -field.*

We briefly describe the plan of the proof. To show that whenever F is a C -field we have $\mathcal{G}_F = G_F^{\{3\}}$, we use a classification of C -fields [32, 1.1] and work case-by-case to show that $F^{\{3\}} = F^{(3)}$. To prove the converse, we note that by [32, 1.9], if F is not a C -field, then there exists a binary quadratic form of a certain type over F , which implies the existence of a certain permanent cycle in $E_\infty^{1,1}$, the mod 2 Lyndon-Hochschild-Serre spectral sequence for the group extension $1 \rightarrow \Phi \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1$. An identification of $E_\infty^{1,1}$ using Merkurjev's theorem [23] then shows that $G_F^{[2]}$ must act nontrivially on $G_{F^{(2)}}^{[2]}$, and so it follows that $\mathcal{G}_F \neq G_F^{\{3\}}$.

As mentioned above our proof depends on Ware's classification of C -fields, specifically the following result (recall that $s(F)$ denotes the level of the field F):

Proposition 3.2. *Let F be a C -field. If F is formally real, then F is superpythagorean, while if F is not formally real, $s(F) = 1$ or 2 .*

Remark 3.3. A field F is *pythagorean* if $F^2 + F^2 = F^2$. A formally real field F is *superpythagorean* if it is pythagorean and given any subgroup S of index 2 in \dot{F} such that $-1 \notin F$, S is an ordering of F . For further details, see [22, p. 44].

It is clear from the proposition above that we can prove one implication in Theorem 3.1 by studying three cases: the superpythagorean case, where $s(F) = \infty$, the case $s(F) = 1$, and the case $s(F) = 2$. Indeed, it is a consequence of the following three lemmas.

Lemma 3.4. *If F is a superpythagorean field then $\mathcal{G}_F = G_F^{\{3\}}$.*

Lemma 3.5. *If F is a C -field with $s(F) = 1$ then $\mathcal{G}_F = G_F^{\{3\}}$.*

Lemma 3.6. *If F is a C -field with $s(F) = 2$ then $\mathcal{G}_F = G_F^{\{3\}}$.*

Remark 3.7. The proofs of Lemmas 3.4, 3.5, and 3.6 borrow heavily from the study of the W -groups of the relevant types of fields in [25]. In that paper, the possible W -groups of a C -field were determined, and their structure turns out to depend on the same sort of structural properties of \dot{F}/\dot{F}^2 as are used in the proofs of the following lemmas. To clarify things, we provide a list of the possible W -groups of a C -field.

Proof of Lemma 3.5. First we discuss the possible W -groups of C -fields F with $s(F) = 1$. If $|\dot{F}/\dot{F}^2| = 1$, F is quadratically closed, so $F^{\{3\}} = F^{(2)} = F$ and $\mathcal{G}_F = 1$. If $|\dot{F}/\dot{F}^2| = 2$, then $\mathcal{G}_F = \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$; by [25, 3.12.3] if $\mathcal{G}_F = \mathbb{Z}/2\mathbb{Z}$, then F is euclidean, and in particular -1 is not a square in F , so this case is impossible for a C -field of level 1. Thus we see that if F is a C -field of level 1 and $|\dot{F}/\dot{F}^2| = 1$ or 2 , then $\mathcal{G}_F = 1$ or $\mathbb{Z}/4$, accordingly.

Now let us suppose that $\dot{F}/\dot{F}^2 = \oplus_I \mathbb{Z}/2\mathbb{Z}$, where $|I| \geq 2$. Then from [25, 3.13] we have that the W -group of F is $\prod_I \mathbb{Z}/4\mathbb{Z}$.

So we must prove that when $\mathcal{G}_F = \prod_I \mathbb{Z}/4\mathbb{Z}$, $F^{\{3\}} = F^{(3)}$. As we noted in the first paragraph of this proof, if $I = \emptyset$, this is trivial, so (although it is not logically necessary) we can eliminate the need to keep track of trivial cases by assuming that $|I| \geq 1$.

To show that $F^{\{3\}} = F^{(3)}$ we will show for each $\gamma \in \dot{F}^{(2)}$ that $F^{(2)}(\sqrt{\gamma})/F$ is Galois, so that $F^{\{3\}} \subset F^{(3)}$, and hence $F^{\{3\}} = F^{(3)}$. Our method of demonstrating that $F^{(2)}(\sqrt{\gamma})/F$ is Galois will be to prove something a bit stronger, namely that for each $\gamma \in \dot{F}^{(2)}$ there is an $a \in \dot{F}$ such that $[\gamma] = [\sqrt{a}]$ in $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$. From this it follows that $F^{(2)}(\sqrt{\gamma}) = F^{(2)}(\sqrt[4]{a})$, which is a Galois extension of F since $\sqrt{-1} \in F^{(2)}$.

So, let us take $\gamma \in \dot{F}^{(2)}$ and seek an element $a \in \dot{F}$ such that $[\gamma] = [\sqrt{a}]$ in $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$. Note first that $\gamma \in F(\sqrt{a_1}, \dots, \sqrt{a_n})$ for some elements a_1, \dots, a_n of F , and that we may take $[a_1], \dots, [a_n]$ to be linearly independent in \dot{F}/\dot{F}^2 . Writing $K := F(\sqrt{a_1}, \dots, \sqrt{a_n})$ and using the proof of [25, 3.13], it follows that we may complete our partial basis of \dot{F}/\dot{F}^2 to a basis $\mathcal{B} = \{[a_1], \dots, [a_n]\} \cup \{[a_j] \mid j \in \Omega - \underline{n}\}$ of \dot{F}/\dot{F}^2 with the property that $\tilde{\mathcal{B}} = \{[\sqrt{a_1}], \dots, [\sqrt{a_n}]\} \cup \{[a_j] \mid j \in \Omega - \underline{n}\}$ is a basis of \dot{K}/\dot{K}^2 . (Here Ω is an ordinal number, used as an indexing set, and $\underline{n} := \{1, \dots, n\}$.) Then in \dot{K}/\dot{K}^2 we have

$$\gamma = \prod_{i=1}^n [\sqrt{a_i}]^{\epsilon_i} \cdot \prod_{j \in \Omega - \underline{n}} [a_j]^{\epsilon_j},$$

where $\epsilon_i = 0$ or 1 and all but finitely many terms in the product are 1 , so that

$$[\gamma] = \left[\sqrt{\prod_{i=1}^n [a_i]^{\epsilon_i} \cdot \prod_{j \in \Omega - \underline{n}} [a_j]^{2\epsilon_j}} \right]$$

in \dot{K}/\dot{K}^2 , and hence also in $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$ since $K \subset F^{(2)}$.

This completes the proof that $F^{\{3\}} = F^{(3)}$ for C -fields of level 1 . \square

The proofs of Lemmas 3.4 and 3.6 are quite similar to the proof above and so we give a somewhat abbreviated presentation.

Proof of Lemma 3.4. We follow the same plan: we show that for each $\gamma \in \dot{F}^{(2)}$ there is an $a \in \dot{F}$ such that $[\gamma] = [\sqrt{a}]$ in $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$. It follows that for any $\gamma \in \dot{F}^{(2)}$ that $F^{(2)}(\sqrt{\gamma})/F$ is Galois, so that $F^{\{3\}} = F^{(3)}$. The key point is again that there is a basis of \dot{F}/\dot{F}^2 of the form $\{[-1]\} \cup \{[a_i] \mid i \in I\}$ such that the set $\{[\sqrt{a_i}] \mid i \in I\}$ is a basis of $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$.

Remark 3.8. The failure to list $\sqrt{-1}$ as a basis element of $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$ is not a mistake. If L is a field of characteristic not equal to 2 , then $\sqrt{2} \in L^{(2)}$ and $\sqrt{-1} \in L^{(2)}$, so that $\sqrt[4]{-1} = \frac{\sqrt{2}}{2}(1 + \sqrt{-1}) \in L^{(2)}$, i.e. $\sqrt{-1}$ is a square in $L^{(2)}$.

The proof that $F^{\{3\}} = F^{(3)}$ is then completed by mimicking the details at the end of the proof of Lemma 3.5. \square

The proof of Lemma 3.6 will involve an appeal to the following fact, whose proof can be found in [32].

Lemma 3.9. *Let K/F be a Galois extension and a an element of \dot{K} . Then the extension $K(\sqrt{a})/F$ is Galois if and only if $\sigma(a) \cdot a$ is a square in \dot{K} for every $\sigma \in \text{Gal}(K/F)$.*

Proof of Lemma 3.6. As in the proof of Lemma 3.5, it is enough to show that for each $\gamma \in \dot{F}^{(2)}$ that $F^{(2)}(\sqrt{\gamma})/F$ is Galois. Since in the case at hand, F is a C -field of level 2, by [25, p. 527], we have a basis for $\dot{F}^{(2)}/(\dot{F}^{(2)})^2$ of the form $\tilde{\mathcal{B}} = \{[y], [\sqrt{a_i}] \mid i \in \Omega\}$, where $y \in F(\sqrt{-1})$ and such that $\mathcal{B} = \{[-1], [a_i] \mid i \in \Omega\}$ is a basis of \dot{F}/\dot{F}^2 .

We claim that in order to show that for every $\gamma \in \dot{F}^{(2)}$, $F^{(2)}(\sqrt{\gamma})/F$ is Galois, it suffices to prove this fact for γ in $\tilde{\mathcal{B}}$.

Proof of claim. Suppose $\gamma = \gamma_1 \cdots \gamma_s$, where $\gamma_i \in \tilde{\mathcal{B}}$. To show that $F^{(2)}(\sqrt{\gamma})/F$ is Galois, by Lemma 3.9 it is enough to show that $\sigma(\gamma) \cdot \gamma$ is a square in $F^{(2)}$ for each $\sigma \in G_F^{[2]}$. But if this is true for each $\gamma_i \in \tilde{\mathcal{B}}$, the calculation

$$\sigma(\gamma) \cdot \gamma = \sigma(\gamma_1 \cdots \gamma_s) \cdot \gamma_1 \cdots \gamma_s = \sigma(\gamma_1) \cdot \gamma_1 \cdots \sigma(\gamma_s) \cdot \gamma_s = x_1^2 \cdots x_s^2$$

shows that it is also true for γ .

So we must only show that for each $\gamma \in \tilde{\mathcal{B}}$ that $F^{(2)}(\sqrt{\gamma})/F$ is Galois. If $\gamma = \sqrt{a_i}$, this follows from the fact that $\sqrt{-1} \in F^{(2)}$, while if we take $\gamma = y$, we may use Lemma 3.9, and note that $\sigma(y) \cdot y$ is either y^2 , which is obviously a square, or $N_{F(\sqrt{-1})/F}(y)$, which is an element of F and therefore a square in $F^{(2)}$. \square

For the other half of the proof we will use group cohomology and a result due to Merkurjev. Our basic goal is the identification of the $E_\infty^{1,1}$ -term of the Lyndon-Hochschild-Serre spectral sequence for the group extension

$$1 \rightarrow \Phi \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1$$

in terms of information in the V -group of F , $G_F^{\{3\}}$. To describe this information, we first introduce some notation.

Recall that there is a group extension $1 \rightarrow A \rightarrow G_F^{\{3\}} \rightarrow G_F^{[2]} \rightarrow 1$ where the Pontrjagin dual of A is isomorphic to $F^{(2)}/(F^{(2)})^2$. This elementary abelian 2-group is a $G_F^{[2]}$ -module.

Let $J := F^{(2)}/(F^{(2)})^2$, regarded as a $G_F^{[2]}$ -module. The socle series of J ,

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J$$

has finite length l (i.e. $J_l = J$) if \dot{F}/\dot{F}^2 is finite. This length is an invariant of the field, which we will write as $l(F)$. In this socle series, $J_1 = J^{G_F^{[2]}}$, $J_2/J_1 = (J/J_1)^{G_F^{[2]}}$, \dots , $J_{i+1}/J_i = (J/J_i)^{G_F^{[2]}}$, etc. We will loosely refer to this as the length of the field F .

Now we can state our key result

Proposition 3.10. *In the mod 2 Lyndon–Hochschild–Serre spectral sequence for the group extension $1 \rightarrow \Phi \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1$, we have that $E_\infty^{1,1} \approx J_2/J_1$.*

Note that the group extension is one for the W -group, but that the group J_2/J_1 comes from the V -group. Thus, this fact is more subtle than it looks. In fact its proof involves the use of Merkurjev’s theorem in an essential way.

Consider the commutative diagram of extensions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & G^{(2)} & \longrightarrow & G_q & \longrightarrow & G_F^{[2]} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \Phi & \longrightarrow & \mathcal{G}_F & \longrightarrow & G_F^{[2]} \longrightarrow 1. \end{array}$$

We shall denote the spectral sequences for the mod 2 cohomology of \mathcal{G}_F and G_q by E and \bar{E} respectively. There is an induced map from $E_2^{1,1}$ to $\bar{E}_2^{1,1}$, for which we will write γ .

Lemma 3.11. $\bar{d}_2^{1,1}$ is injective.

Proof. By Merkurjev’s theorem, the inflation map from $H^2(G_F^{[2]})$ to $H^2(G_q)$ is surjective, so $\bar{E}_\infty^{1,1} = 0$, but this is just the kernel of $\bar{d}_2^{1,1}$. \square

Lemma 3.12. $\ker(\gamma) = \ker(d_2^{1,1})$.

Proof. This follows by considering the following commutative diagram and applying the previous lemma.

$$\begin{array}{ccc} \bar{E}_2^{1,1} & \xrightarrow{\bar{d}_2^{1,1}} & H^3(G_F^{[2]}) \\ \gamma \uparrow & & \parallel \\ E_2^{1,1} & \xrightarrow{d_2^{1,1}} & H^3(G_F^{[2]}) \end{array}$$

\square

Lemma 3.13. $\ker(\gamma) = J_2/J_1$

Proof. We have $\bar{E}_2^{1,1} = H^1(G_F^{[2]}, H^1(G^{(2)}))$ and $E_2^{1,1} = H^1(G_F^{[2]}, H^1(\Phi))$. Since $J_1 \approx H^1(\Phi)$ and $J \approx H^1(G^{(2)})$, the result follows from considering the long exact sequence in cohomology associated to the short exact sequence of coefficients $J_1 \hookrightarrow J \twoheadrightarrow J/J_1$, since γ is just the map $H^1(G_F^{[2]}, J_1) \rightarrow H^1(G_F^{[2]}, J)$. \square

Now the proof of Proposition 3.10 is merely a matter of stringing the lemmas together.

Proof of Proposition 3.10. Note that $E_\infty^{1,1} = E_3^{1,1} = \ker(d_2^{1,1})$, as there are no further differentials in this part of the spectral sequence. By Lemma 3.12, $\ker(d_2^{1,1}) = \ker(\gamma)$, which is J_2/J_1 by Lemma 3.13. \square

We can now show that if F is not a C -field, then the V -group of F is *not* the W -group of F . The proof of this fact is based on an examination of the spectral sequence for the cohomology of the W -group (3.15) and an identification of part of the E_∞ -term of that spectral sequence with information contained in the V -group (Lemma 3.14).

Lemma 3.14. *The following conditions are equivalent:*

1. $J = J_1$
2. $F^{\{3\}} = F^{(3)}$
3. $J_2 = J_1$.

Proof. $1 \Leftrightarrow 2$. We need the fact that an extension $F^{(2)}(\sqrt{\gamma})/F$ is Galois if and only if $[\gamma] \in J_1$. This follows from Lemma 3.9.

We have shown that $F^{(3)}$ is obtained from $F^{(2)}$ by adjoining the square roots of elements in J_1 , while $F^{\{3\}}$ is obtained by adjoining the square roots of elements in J . Therefore we have $F^{\{3\}} = F^{(3)}$ if and only if $J = J_1$.

$1 \Leftrightarrow 3$. To show that $J_2 = J_1$ implies $J = J_1$ we note that for any element $j \in J$, the orbit of j under the action of $G_F^{[2]}$ is finite. This is because we may choose a representative of j , which is an element of $\tilde{F}^{(2)}$, hence algebraic over F , so that any image of this element under the action of the Galois group satisfies the same minimal polynomial. Therefore, if $\bar{j} \in J/J_1$, the $G_F^{[2]}$ submodule generated by \bar{j} is finite and has a fixed point by the usual counting argument. Thus if $J/J_1 \neq 0$, $J_2 \neq J_1$, which is the contrapositive of the desired statement. The converse is trivial. \square

Now we turn to the study of the spectral sequence for the group extension

$$1 \rightarrow \Phi \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1.$$

We will show that if F is not a C -field, then there are classes in the spectral sequence that survive to $E_\infty^{1,1}$. More precisely we will prove:

Proposition 3.15. *Suppose F is not a C -field. Then in the spectral sequence for the cohomology of \mathcal{G}_F , $E_\infty^{1,1} \neq 0$.*

In the proof of this proposition we will need the following lemma which follows from [21, theorem 2.7, p. 58] and [26, p. 255].

Lemma 3.16. *Let L be a field and a, b elements of \dot{L} . Writing $[a]$ and $[b]$ for the classes of a and b in \dot{L}/\dot{L}^2 , or equivalently in $H^2(\text{Gal}(L_q/L); \mathbb{F}_2)$, we have $[a] \cup [b] = 0$ if and only if $ax^2 + by^2 - z^2 = 0$ has a nontrivial solution $(x, y, z) \in L^3$.*

Remark 3.17. More precisely, we need the fact that this lemma remains true if the Galois group of the quadratic closure L_q is replaced by the W -group \mathcal{G}_L . But this is true by the construction of the W -group and Merkurjev's theorem. See [2, 3.14].

Proof of Proposition 3.15. Since F is not a C -field, by [32, 1.9] there exists an anisotropic binary form B over F such that the set D of values of B , regarded as a subgroup of \dot{F}/\dot{F}^2 , has at least three elements. (As \dot{F}/\dot{F}^2 is an elementary abelian 2-group, so are any of its subgroups, so “ D has at least three elements” immediately implies “ D has at least four elements”. The fact that D is a subgroup follows from the identity $(1 - ax^2)(1 - ay^2) = (1 + axy)^2 - a(x + y)^2$.) We may assume that $B = x^2 - ay^2$ for some $a \in \dot{F}$ by an appropriate transformation. Because B is anisotropic, we know that $[a] \neq [1]$ in \dot{F}/\dot{F}^2 .

Since $|D| \geq 4$, there exist elements a_2, a_3 in D which are linearly independent over \mathbb{F}_2 . The statement “ a_2 is a value of B in \dot{F}/\dot{F}^2 ” means that there exist x, y , and z in F such that $a_2x^2 + ay^2 - z^2 = 0$. By Lemma 3.16 we see that $[a][a_2] = 0 \in H^2(\mathcal{G}_F)$, and similarly that $[a][a_3] = 0$.

This information allows us to construct directly a nonzero element in the spectral sequence for $H^*(\mathcal{G}_F)$. Because the relations $[a][a_2] = 0$ and $[a][a_3] = 0$ exist in $H^*(\mathcal{G}_F)$ but not in $H^*(G_F^{[2]})$, there exist z_2, z_3 in $H^1(\Phi)$ such that $d_2^{0,1}(z_2) = [a][a_2]$ and $d_2^{0,1}(z_3) = [a][a_3]$. Then, setting $\lambda = [a_2] \otimes z_3 + [a_3] \otimes z_2 \in E_2^{1,1} = H^1(G_F^{[2]}) \otimes H^1(\Phi)$, we can compute

$$d_2^{1,1}(\lambda) = [a_2] \cdot d_2^{0,1}(z_3) + [a_3] \cdot d_2^{0,1}(z_2) = [a_2][a][a_3] + [a_3][a][a_2] = 0.$$

Thus λ survives to $E_3^{1,1}$ and hence to $E_\infty^{1,1}$ by its position in the spectral sequence. \square

The results above immediately imply the converse implication in the statement of 3.1 and so the proof is complete.

4. SURJECTIVITY IN MERKURJEV'S THEOREM AND AN ANALOGUE OF HILBERT'S THEOREM 90

We have seen how methods from group cohomology can be used to characterize certain fields. In this section we make an explicit analysis of the kernel of the cohomological differential d_2 used previously and interpret the result in field-theoretic terms. From this we obtain a result which is reminiscent of Hilbert's Theorem 90.

Let $\{[a_i] \mid i \in I\}$ be a basis for \dot{F}/\dot{F}^2 , which we identify with $H^1(G_F^{[2]})$. Let $\{\sigma_i \mid i \in I\}$ be a minimal set of generators of $G_F^{[2]}$ such that $\sigma_i(\sqrt{a_j})/\sqrt{a_j} = (-1)^{\delta_{i,j}}$, where $i, j \in I$ and $\delta_{i,j}$ is the Kronecker delta function. One may view the a_i as a dual basis to the σ_i .

We are now able to state the result:

Theorem 4.1. *Let $\{[j_{i_k}] \mid 1 \leq k \leq l\}$ be a finite set of elements of J_1 . Then*

$$(1) \quad \sum_{k=0}^l [a_{i_k}] d_2^{0,1}([j_{i_k}]) = 0$$

if and only if there exists an element $[j] \in J_2$ such that

1. $[\sigma_{i_k}(j)/j] = [j_{i_k}]$ for each $k \in \{1, \dots, l\}$ and
2. $[\sigma_u(j)] = [j]$ for each $u \notin \{i_1, \dots, i_l\}$.

It is worth noting that one implication of the theorem (the “if” part) can be proved directly, without the use of Merkurjev’s theorem or spectral sequences. We indicate how this can be done in an appendix (§7).

Proof of Theorem 4.1. Recall that we have the exact sequence

$$0 \rightarrow J_2/J_1 \rightarrow H^1(G_F^{[2]}, J_1) \rightarrow H^1(G_F^{[2]}, J)$$

arising from the long exact sequence in cohomology associated to the short exact sequence of coefficients $J_1 \hookrightarrow J \twoheadrightarrow J/J_1$. Note that since the action of $G_F^{[2]}$ on the coefficient group J_1 is trivial, we may identify $H^1(G_F^{[2]}, J_1)$ with $\text{Hom}(G_F^{[2]}, J_1)$. In other words $J_2/J_1 \cong \text{Ker}(\text{Hom}(G_F^{[2]}, J_1) \rightarrow H^1(G_F^{[2]}, J))$. We shall make this isomorphism explicit. Let $[j] \in J_2/J_1$. Then we can associate to this $[j]$ the function $f_{[j]} : G_F^{[2]} \rightarrow J_1$ which is given by the formula:

$$f_{[j]}(\sigma) = \left[\frac{\sigma(j)}{j} \right] \in J_1.$$

It follows from the definition of socle series that $[\sigma(j)/j] \in J_1$. The connecting homomorphism $J_2/J_1 \rightarrow H^1(G_F^{[2]}, J_1)$ is given by $[j] \mapsto f_{[j]}$. In other words, we have a one-to-one correspondence $J_2/J_1 \rightarrow \text{Ker}(\text{Hom}(G_F^{[2]}, J_1) \rightarrow H^1(G_F^{[2]}, J))$ given by $[j] \mapsto f_{[j]}$. Using the basis $\{a_i\}$ introduced earlier each function $f_{[j]}$ can be written as

$$f_{[j]} = [a_{i_1}] \otimes [j_{i_1}] + \dots + [a_{i_l}] \otimes [j_{i_l}] \in H^1(G_F^{[2]}) \otimes J_1,$$

simply because $\text{Hom}(G_F^{[2]}, J_1) = \text{Hom}(G_F^{[2]}, \mathbb{Z}/2\mathbb{Z}) \otimes J_1$. Observe that our function $f_{[j]}$ has the values:

$$f_{[j]}(\sigma_{i_1}) = [j_{i_1}], \dots, f_{[j]}(\sigma_{i_k}) = [j_{i_k}]$$

and

$$f_{[j]}(\sigma_l) = [1], \quad \text{for each } l \neq i_1, \dots, i_k.$$

(Recall that each continuous homomorphism $G_F^{[2]} \rightarrow J_1$ has only finitely many values by definition of the Krull topology on $G_F^{[2]}$.)

Combining the above with the fact that $J_2/J_1 \cong E_\infty^{1,1}$ (3.10) and the definition of $E_\infty^{1,1} = \ker d_2^{1,1}$ completes the proof. □

Example 4.2. Let F be a field such that $\mathcal{G}_F \cong W(2)$. In this case G_q is the free pro-2-group on two generators, and as we saw before, the V -group is a quotient expressible as an associated “abelianized extension”.

We can choose a basis $\{[a_1], [a_2]\}$ of \dot{F}/\dot{F}^2 and we can choose as our generating k -invariants of $W(2)$ the following elements of $H^2(G_F^{[2]})$:

$$q_1 = d_2^{0,1}([j_1]) = [a_1][a_1], \quad q_2 = d_2^{0,1}([j_2]) = [a_2][a_2], \quad q_3 = d_2^{0,1}([j_3]) = [a_1][a_2].$$

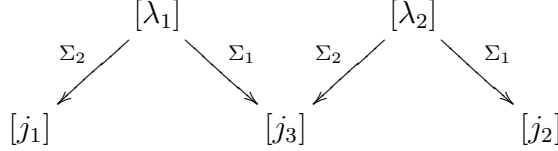
Thus $J_1 = \langle [j_1], [j_2], [j_3] \rangle \subset J$.

Set also $\lambda_1 = [a_1] \otimes [j_3] + [a_2] \otimes [j_1] \in E_2^{1,1}$, $\lambda_2 = [a_1] \otimes [j_2] + [a_2] \otimes [j_3] \in E_2^{1,1}$.

We see that $\langle \lambda_1, \lambda_2 \rangle = \ker d_2^{1,1}$ and therefore λ_1, λ_2 form a basis of $E_\infty^{1,1}$. We will identify λ_1, λ_2 with elements $[\lambda_1], [\lambda_2] \in J_2/J_1$ (here we are using 3.10).

We pick as usual generators σ_1, σ_2 for $G_F^{[2]}$. Then we have $\sigma_1([\lambda_1]) = [\lambda_1][j_3]$ and $\sigma_2([\lambda_1]) = [\lambda_1][j_1]$ similarly the expression for λ_2 as an element of $E^{1,1}$ determines the action of $G_F^{[2]}$ on $[\lambda_2]$: $\sigma_2[\lambda_2] = [j_3][\lambda_2]$ and $\sigma_1[\lambda_2] = [\lambda_2][j_2]$.

It is convenient to picture our $G_F^{[2]}$ -module J_2 as follows:



Here $\Sigma_2[\lambda_2] := (\sigma_2 - 1)[\lambda_2] := \sigma_2[\lambda_2]/[\lambda_2]$, etc.

As mentioned above, G_q is the free pro-2-group on two generators (see [28, 4.3, pp. 33,34]). Recall that the well-known formula, due to Schreier, on the number of any minimal set of generators of any subgroup of finite index of a free group has a pro- p -analogue [19, p. 49, 6.3]. Set $G^{(2)} := \text{Gal}(F_q/F^{(2)})$. Then $G^{(2)}$ is an open subgroup of G_q , since here $|\dot{F}/\dot{F}^2| = 4 < \infty$. Then by the aforementioned formula we have

$$\dim_{\mathbb{F}_2} H^1(G^{(2)}) = 4(\dim H^1(G_q) - 1) + 1.$$

Hence we see that $\dim_{\mathbb{F}_2} H^1(G^{(2)}) = 5$. From Pontrjagin duality we know that

$$J = \dot{F}^{(2)}/(\dot{F}^{(2)})^2 \cong H^1(G^{(2)}).$$

Thus $\dim_{\mathbb{F}_2} J = 5$. Since $\dim_{\mathbb{F}_2} J_1 = 3$ and $\dim_{\mathbb{F}_2} J_2/J_1 = 2$ we see that $l(J) = 2$. This means $J_2 = J$.

Therefore we have the following extension

$$1 \rightarrow A \rightarrow G_F^{\{3\}} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

Our group $A \cong (\mathbb{Z}/2\mathbb{Z})^5 = \langle \hat{\sigma}_1^2, \hat{\sigma}_2^2, [\hat{\sigma}_1, \hat{\sigma}_2], [\hat{\sigma}_1^2, \hat{\sigma}_2], [\hat{\sigma}_2^2, \hat{\sigma}_1] \rangle$. J is just the dual of A . This means $J \cong H^1(A) \cong \Omega^{-2}(\mathbb{F}_2)$. Observe also that our basis $\{[j_1], [j_2], [j_3], [\lambda_1], [\lambda_2]\}$ is dual to the basis of A given above with respect to the Kummer pairing. Thus we may write

$$[j_1] = (\hat{\sigma}_1^2)^*, \quad [j_2] = (\hat{\sigma}_2^2)^*, \quad [j_3] = ([\sigma_1, \sigma_2])^*,$$

$$[\lambda_1] = ([\sigma_1^2, \sigma_2])^*, \quad [\lambda_2] = ([\sigma_2^2, \sigma_1])^*.$$

It is not hard to use the description above to make explicit the existence of a central extension $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow V(2) \rightarrow W(2) \rightarrow 1$.

5. LENGTH AND COHOMOLOGICAL DIMENSION

We turn now to the case where \dot{F}/\dot{F}^2 is finite, so the order of this group is 2^n for some n . In this case J is a finite-dimensional $G_F^{[2]}$ -module. Therefore its socle series $J_1 \subset J_2 \subset \cdots \subset J_l$ has length l for some $l \in \mathbb{N}$. This number is an interesting field-theoretic invariant and there are conjectural connections between $l(F)$ and the cohomological dimension of G_q . More generally, for $G_F^{[2]}$ -modules M , we will write $l(M)$ for the length of the socle series of M , provided that this length is finite.

If G is a profinite group, its cohomological dimension $\text{cd}(G)$ is defined as $\sup \text{cd}_p(G)$ where $\text{cd}_p(G)$ is the smallest integer n such that the p -primary component of $H^q(G, A)$ is zero for all discrete, torsion G -modules A and all integers $q > n$ (see [28, 3.1]). Recall that if G is a pro-2-group then $\text{cd}(G) = \text{cd}_2(G) \leq n$ iff $H^{n+1}(G, \mathbb{F}_2) = 0$ (see [28, p. 27]).

One of the basic connections we wish to consider appears in [17, 6.2.8]:

Question 5.1. For which non-formally real fields F with $|\dot{F}/\dot{F}^2| = 2^n$ is it true that $l(F) + \text{cd}(G_q) = n + 1$?

We shall provide some examples of classes of fields for which equality holds in the equation above. In the remainder of this section we show that equality holds in the following cases: first, if F (in addition to satisfying the hypotheses of the question) is also a C -field, second, if $\mathcal{G}_F \approx W(n)$, and third, if F is a local field.

Proposition 5.2. *Let F be a C -field which is not formally real, and suppose in addition that $|\dot{F}/\dot{F}^2| = 2^n$. Then $l(F) + \text{cd}(G_q) = n + 1$.*

Proof. We have shown that for C -fields, $\mathcal{G}_F = G_F^{\{3\}}$, so that $J = J_1$ and $l(F) = 1$. So we must show that $\text{cd}(G_q) = n$. From the isomorphism between Galois cohomology and the Witt ring given by the Milnor conjecture, it is enough to show that $I^n F / I^{n+1} F \neq 0$ and that $I^{n+1} F / I^{n+2} F = 0$, where $I^n F$ is the n -th power of the fundamental ideal in the Witt ring. Ware [32] has shown for fields satisfying our hypotheses that $I^n F \neq 0$, while it follows from a result of Kneser [21, p. 317] and the basic theory of Pfister forms that $I^{n+1} F = 0$ in this case. \square

We shall show in what follows that the $G_F^{[2]}$ -module J can be studied in a very concrete way. In the following examples we will use the identification of J_2/J_1 as $E_\infty^{1,1}$ to calculate $\dim_{\mathbb{F}_2}(J_2/J_1)$ in some interesting special cases.

Example 5.3. Suppose that $\mathcal{G}_F = W(n)$. Then it follows from [2, 6.12] that $\dim_{\mathbb{F}_2}(E_\infty^{1,1}) = n(n+1)(n-1)/3$. In fact in this case the group G_q is a free pro-2-group, which means that it has

cohomological dimension equal to one. In this case we have identified the module J with the $G_F^{[2]}$ module $\Omega^{-2}(\mathbb{F}_2)$, which is known to have a socle series of length n . Hence $l(F) + \text{cd}(G_q) = n + 1$, as asserted above.

Remark 5.4. To see that the $l(\Omega^{-2}2(\mathbb{F}_2)) = n$, one can use Remark 5.13 to see that it is enough to prove the surjectivity of $N_{F^{(2)}/F}$. This surjectivity follows by repeatedly applying surjectivity for quadratic extensions K which connect F to $F^{(2)}$. The surjectivity for quadratic extensions follows from the fact that each quaternion algebra defined over K will split over K . To see that this last assertion is true, note that $G_q(F)$ is free, and hence that $G_q(K) \subset G_q(F)$ is also free.

It seems natural to explore the situation for 2-dimensional groups. We recall

Definition 5.5. A Demuškin group (at the prime 2) is a pro-2-group G which is a two-dimensional Poincaré duality group, i.e. $H^2(G, \mathbb{F}_2) = \mathbb{F}_2$, and the cup product

$$H^1(G, \mathbb{F}_2) \times H^1(G, \mathbb{F}_2) \rightarrow H^2(G, \mathbb{F}_2)$$

is a perfect pairing.

Example 5.6. Suppose that F is a local field and $|\dot{F}/\dot{F}^2| = 2^n$. Then G_q is a Demuškin group (see [28, 4.5], and in particular we have $\text{cd } G_q = 2$ and $\dim_{\mathbb{F}_2} H^2(G_q) = 1$. It follows from this and [2, 3.12] that in the extension $1 \rightarrow \Phi \rightarrow \mathcal{G}_F \rightarrow G_F^{[2]} \rightarrow 1$ we have $\dim_{\mathbb{F}_2} \Phi = \binom{n+1}{2} - 1$. We can now calculate $\dim_{\mathbb{F}_2} E_2^{1,1} = n \cdot (\binom{n+1}{2} - 1)$ and $\dim_{\mathbb{F}_2} E_2^{3,0} = n(n+1)(n+2)/6$; since $H^3(G_q) = 0$, it follows that $d_2^{1,1}$ is surjective and that $\dim_{\mathbb{F}_2}(E_\infty^{1,1}) = n(n-2)(n+2)/3 = \dim_{\mathbb{F}_2} J_2/J_1$. To make this example more concrete, we note that this calculation of course applies to the special cases $F = \mathbb{Q}_p$ (p an odd prime), $F = \mathbb{Q}_2$, and $F = \mathbb{Q}_2(\sqrt{2})$. In the case $F = \mathbb{Q}_p$, our calculations give $\dim_{\mathbb{F}_2} J_2/J_1 = 0$, i.e. $J_1 = J_2$. Another way to phrase this fact is to say that \mathbb{Q}_p is a C -field. In the case $F = \mathbb{Q}_2$, we obtain $\dim_{\mathbb{F}_2} J_2/J_1 = 5$, while in the case $F = \mathbb{Q}_2(\sqrt{2})$, $\dim_{\mathbb{F}_2} J_2/J_1 = 16$.

We have thus calculated $|J_1| = |\Phi^*|$ (here Φ^* is the Pontrjagin dual of Φ) as well as $|J_2/J_1|$ for the case of a local field; in fact our calculations are valid for any field F such that G_q is a Demuškin group.

Let G be a Demuškin group and let H be an open subgroup of G . Let $r_G = \dim_{\mathbb{F}_2} H^1(G)$ and $r_H = \dim_{\mathbb{F}_2} H^1(H)$. Then one can verify (see [28, p. 44, ex. 6]) that

$$r_H - 2 = [G : H](r_G - 2).$$

Conversely, this property characterises Demuškin groups (see [15]). We use this characterization to prove

Proposition 5.7. *If G_q is a Demuškin group and $|\dot{F}/\dot{F}^2| = 2^n$, then*

$$\dim_{\mathbb{F}_2} J = 2^n(n-2) + 2$$

Proof. Suppose that $|\dot{F}/\dot{F}^2| = 2^n$. Let $H = G_F^{(2)} := \text{Gal}(F_q/F^{(2)})$. Then H is an open subgroup of G_q and $[G_q : H] = 2^n$. Therefore from the above we obtain $r_H = 2^n(n-2) + 2$. However from Kummer theory we know that $r_H = \dim_{\mathbb{F}_2} \left(\dot{F}^{(2)} / (\dot{F}^{(2)})^2 \right) = \dim_{\mathbb{F}_2} J$. \square

To further our understanding of the $G_F^{[2]}$ -module J we look again at some of the examples above:

Example 5.8. Let $F = \mathbb{Q}_2$. Then $\dot{F}/\dot{F}^2 = \langle [-1], [2], [5] \rangle$, so $n = 3$ and $\dim_{\mathbb{F}_2} J = 2^3(3-2) + 2 = 10$. Notice that this is the same as $\dim_{\mathbb{F}_2} \Phi + \dim_{\mathbb{F}_2} J_2/J_1 = 6 - 1 + 5$. In particular $l(\mathbb{Q}_2) = 2$, and so the desired equality holds in this case.

Example 5.9. Let $F = \mathbb{Q}_2(\sqrt{2})$, so $n = 4$. Then $\dim_{\mathbb{F}_2} J = 16 \cdot 2 + 2 = 34$. We know already 5.6 that $\dim_{\mathbb{F}_2} \Phi = \binom{5}{2} - 1 = 9$ and that $\dim_{\mathbb{F}_2} J_2/J_1 = 16$. If the equality holds in this case, then we have $l(F) = 5 - 2 = 3$, so that $J = J_3$ and $\dim_{\mathbb{F}_2} J_3/J_2 = \dim_{\mathbb{F}_2} J/J_2 = 34 - 25 = 9$.

Example 5.10. Suppose that F is a local field with $|\dot{F}/\dot{F}^2| = 2^n$. Then we can compute $\dim_{\mathbb{F}_2}(J/J_2)$:

$$\begin{aligned} \dim_{\mathbb{F}_2}(J/J_2) &= \dim_{\mathbb{F}_2} J - \dim_{\mathbb{F}_2} J_2/J_1 - \dim_{\mathbb{F}_2} J_1 \\ &= 2^n(n-2) + 2 - \frac{n(n-2)(n+2)}{3} - \binom{n+1}{2} + 1 \\ &= 2^n(n-2) + 3 - \frac{n(n-2)(n+2)}{3} - \binom{n+1}{2}. \end{aligned}$$

We will now make a more in-depth analysis of the module J . We recall the following basic fact from local class field theory:

Lemma 5.11. *Let F be a local field such that $|\dot{F}/\dot{F}^2| = 2^n$. If $N_{F^{(2)}/F} : \dot{F}^{(2)} \rightarrow \dot{F}$ is the usual norm map then its image lies in \dot{F}^2 . If $F \subset K \subset F^{(2)}$ is a proper quadratic extension of F , then the image of $N_{F^{(2)}/K} : \dot{F}^{(2)} \rightarrow \dot{K}$ does not lie in \dot{K}^2 .*

Proof. From local class field theory (see [14]) we know that there is a natural isomorphism $\text{Gal}(F^{(2)}/L) \cong \dot{L} / \text{Im } N_{F^{(2)}/L}$ for any intermediate extension $F \subset L \subset F^{(2)}$. If $L = F$ we obtain the first statement. For $L = K$ we observe (see [21], page 202) that for any proper quadratic extension K of F , $|\dot{K}/\dot{K}^2| \geq |\dot{F}/\dot{F}^2|$; however $|\text{Gal}(F^{(2)}/K)| < |\text{Gal}(F^{(2)}/F)| = |G_F^{[2]}|$. We conclude that $\text{Im } N_{F^{(2)}/K}$ cannot be contained in \dot{K}^2 . \square

This relates to our length invariant via the following

Proposition 5.12. *Let F be a field such that $|\dot{F}/\dot{F}^2| = 2^n$. Then, in the socle series $J_1 \subset J_2 \subset \dots \subset J_{l(F)}$, the submodule J_{n-1} is equal to the kernel of the homomorphism $N : J \rightarrow \dot{F}/\dot{F}^2$ induced by the norm map above.*

Remark 5.13. The proposition above can be extended to a characterization of the complete socle series $\{J_i\}$; a proof by induction, using [16, p. 133], which we omit, shows that $J_i = \{[j] \in J \mid N_{F^{(2)}/L}(j) = [1] \in \dot{L}/\dot{L}^2 \text{ for all } L \text{ with } [F^{(2)} : L] = 2^{i+1}\}$.

Proof of Proposition 5.12. Let $F \subset L \subset F^{(2)}$ denote an extension of F such that $[F^{(2)} : F] = 2^k$. Consider $\text{Gal}(F^{(2)}/L)$; then by restriction J will also be a module over this group. Denote the usual module-theoretic norm map by $T_L : J \rightarrow J$. From the definition of the socle series for J it follows that $J_{k-1} = \cap \ker T_L$, where the intersection is taken over all extensions as above, of co-degree 2^k . Let L denote a proper quadratic extension of F . The norm map T_L is induced from the composition of the field-theoretic norm map $N_{F^{(2)}/L} : \dot{F}^{(2)} \rightarrow \dot{L}$ with the maps $c_L : \dot{L} \rightarrow \dot{L}/\dot{L}^2$ and $i_{L/F} : \dot{L}/\dot{L}^2 \rightarrow J$ (the latter induced by the inclusion $L \subset F^{(2)}$). Hence if $[j] \in J$, we see that $[j] \in \ker T_L$ if and only if $c_L(N_{F^{(2)}/L}(j)) \in \ker i_{L/F}$. However from Kummer theory (see [5], page 21, theorem 3) we have that $\ker i_{L/F} = \dot{F}\dot{L}^2/\dot{L}^2$, and so $T_L(j) = 0$ if and only if $N_{F^{(2)}/L}(j) \in \dot{F}\dot{L}^2$. However, from [21], theorem 3.4, pp. 202, we know that $\mu \in \dot{L}$ belongs to $\dot{F}\dot{L}^2$ if and only if $N_{L/F}(\mu) \in \dot{F}^2$. Therefore, from the transitivity property of the norm we see that $T_L([j]) = 0$ if and only if $N_{F^{(2)}/F}(j) \in \dot{F}^2$. \square

From this (5.12) we obtain

Corollary 5.14. *Let F denote a local field with $|\dot{F}/\dot{F}^2| = 2^n$; then $l(F) = n - 1$.*

Proof. From our description of J_{n-1} and the triviality of the map induced by the norm, we infer that $J = J_{n-1}$, whence $l(F) \leq n - 1$. Taking a quadratic extension $F \subset K \subset F^{(2)}$ we see that in the socle series for J as a $\text{Gal}(F^{(2)}/K)$ -module, $J_{n-2} \neq J$. Hence $l(F) = n - 1$. \square

This can be restated as

Theorem 5.15. *Let F denote a local field with $|\dot{F}/\dot{F}^2| = 2^n$. Then $l(F) + \text{cd}(G_q) = n + 1$.*

It seems rather complicated to verify the relationship above for other types of fields. We briefly outline a more cohomological approach.

Given any finitely generated $\mathbb{F}_2 G_F^{[2]}$ -module M , there is a minimal power I^t of the augmentation ideal (which is nilpotent) such that $I^t M = 0$; $t = \lambda(M)$ is called the Loewy length of M . Note that if $M \neq 0$, then $1 \leq \lambda(M) \leq n + 1$. Moreover M is a trivial $G_F^{[2]}$ -module if and only if $\lambda(M) = 1$. The length of the socle series of a module is equal to the Loewy length of its dual. Consider the following result, which follows from a theorem due to G. Carlsson ([13]):

Proposition 5.16. *Suppose that H and G are pro-2-groups, which are topologically finitely generated, G of finite (continuous) cohomological dimension k at the prime 2, and which have finite total mod 2 cohomology. Assume in addition that E is an elementary abelian 2-group of rank equal to n , and that*

$$1 \rightarrow H \rightarrow G \rightarrow E \rightarrow 1$$

is an extension. Then it follows that

$$\lambda(H^1(H, \mathbb{F}_2)) + \cdots + \lambda(H^k(H, \mathbb{F}_2)) \geq n.$$

In particular if H and G are 2-dimensional Poincaré duality groups, $H^1(H, \mathbb{F}_2)$ is self-dual, $\lambda(H^1(H, \mathbb{F}_2)) \geq n - 1$, and therefore $l(H^1(H, \mathbb{F}_2)) \geq n - 1$.

As an immediate consequence of the above we obtain a different proof of the inequality $l(F) \geq n - 1$ for fields F such that G_q is Demuškin. As indicated by this cohomological method, the filtration lengths of higher cohomology groups will probably play a role in any generalization of this question to groups of larger cohomological dimension.

6. THE COHOMOLOGY OF $V(2)$

In this section we study the cohomology of the universal V -group $V(2)$ defined previously. We will first describe some basic facts about $V(n)$ expressing it in terms of certain group extensions which will be useful in computing and interpreting cohomology. Our point of view will closely follow the analysis of W -groups made in [2], so we will be brief. Many of the calculations described below can be done partially or entirely using a computer algebra system such as MAGMA [8], so almost all detailed justifications are omitted.

We begin by taking the standard surjection from the free group on n -generators F_n onto the elementary abelian group $E_n = (\mathbb{Z}/2\mathbb{Z})^n$, with kernel the free group on $2^n(n-1)+1$ generators. Associated to this we have the free abelianized extension $1 \rightarrow M \rightarrow X(n) \rightarrow E_n \rightarrow 1$ where M is a $\mathbb{Z}E_n$ lattice of rank as above. It is elementary to verify (see [20] for details) that, as a $\mathbb{Z}E_n$ -module, M is isomorphic to $\Omega^2(\mathbb{Z})$, the second dimension shift of the trivial module. Factoring out the submodule $2M$, we recover an expression for $V(n)$ as an extension of finite groups, $1 \rightarrow M/2M \rightarrow V(n) \rightarrow E_n \rightarrow 1$. This has a simple interpretation: the module $M/2M \cong \Omega^2(\mathbb{F}_2)$ has a unique non zero class in its second cohomology group $H^2(E_n, \Omega^2(\mathbb{F}_2)) = \mathbb{F}_2$; the group $V(n)$ realizes this extension class. Similarly, the group $X(n)$ is a Bieberbach group (see [35]) corresponding to the canonical generator in $H^2(E_n, \Omega^2(\mathbb{Z})) \cong \mathbb{Z}/|E_n|$; it is also expressed as an extension $1 \rightarrow 2M \rightarrow X(n) \rightarrow V(n) \rightarrow 1$.

The comments above show that the group $V(n)$ acts freely on a $2^n(n-1)+1$ -dimensional torus, with quotient the classifying space of $X(n)$. In turn this space can be obtained as the orbit space of a free E_n -action on such a torus. The cohomology of these euclidean space forms is not easy to compute, and we shall see that even the case $n = 2$ poses some interesting technical problems.

We should also point out that from the definition of the V -groups we can also express it as an extension $1 \rightarrow U_r \rightarrow V(n) \rightarrow W(n) \rightarrow 1$ where U_r is a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, $r = 2^n(n-1) - n - \binom{n}{2} + 1$. An important thing to note is that the rank of the largest elementary abelian subgroup in $V(n)$ is precisely equal $r + n + \binom{n}{2}$, or in other words the rank of $V(n)$ differs from the rank of $W(n)$ exactly by the quantity r .

We will now concentrate on the case $n = 2$. Our first result is about the group $X(2)$: we study its mod 2 cohomology via the spectral sequence for the extension $1 \rightarrow \mathbb{Z}^5 \rightarrow X(2) \rightarrow$

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$. The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -module structure of the kernel is known, so we can obtain the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -module structure of the cohomology from the fact that $H^*(\mathbb{Z}^5, \mathbb{F}_2)$ is an exterior algebra. Let $k = \mathbb{F}_2$ denote the coefficient field, then this exterior algebra can be written as a sum of indecomposable $k[\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ -modules as follows:

Degree	Decomposition
0	k
1	$\Omega^{-2}k$
2	$\Omega^1k \oplus \Omega^1k \oplus F$
3	$\Omega^{-1}k \oplus \Omega^{-1}k \oplus F$
4	Ω^2k
5	k

From the table it is straightforward to determine the E_2 -term of the spectral sequence and even the d_2 -differential. It turns out that there are no further possible differentials, so the spectral sequence collapses at E_3 . The next few paragraphs sketch a proof of this fact and record some immediate consequences.

The E_2 -term is not hard to understand as a $H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2)$ -module since all the modules appearing in the table are either free, or dimension shifts of the trivial module. Furthermore, the $d_2^{*,1}$ -differential realizes the cohomology isomorphism $H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \Omega^{-2}k) \approx H^{*+2}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, k)$, since $d_2^{0,1}$ is an isomorphism by the definition of $X(2)$ (specifically, the extension class). By duality it follows that the $d_2^{*,5}$ -differential realizes the cohomology isomorphism $H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, k) \approx H^{*+2}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \Omega^2k)$. Since we know that the classifying space for $X(2)$, and hence its cohomology, is finite-dimensional, it follows that the $d_2^{*,3}$ -differential realizes the cohomology isomorphism $H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \Omega^{-1}k \oplus \Omega^{-1}k) \approx H^{*+2}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \Omega^1k \oplus \Omega^1k)$.

We now note that all this implies

Proposition 6.1. *In the spectral sequence for the extension $1 \rightarrow \mathbb{Z}^5 \rightarrow X(2) \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$, $E_3^{p,q} = 0$ for $p > 1$, and $E_3 = E_\infty$.*

A little more attention to detail gives also

Proposition 6.2. *The Poincaré Series for the cohomology of $X(2)$ is $1 + 2t + 5t^2 + 5t^3 + 2t^4 + t^5$.*

A method similar to that of the previous section could be used to study the cohomology of $V(2)$. However, this group has order only 2^7 , and its cohomology can thus be studied in great detail with a computer. In particular, information on this group is available at Carlson's well-known web site [12].

As the details of the calculation are complicated, we present only an outline of the work necessary to determine by hand calculation the cohomology of $V(2)$. The main reason for outlining this work is to point out the existence of a phenomenon which the careful reader will already have noted in the previous section on $X(2)$.

Now let us sketch the calculation. Studying the cohomology of $V(2)$ via the spectral sequence for the extension $1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^5 \rightarrow V(2) \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow 1$, we must write the symmetric algebra of $\Omega^{-2}k$ as a direct sum of indecomposable modules. If we write P for the direct sum of the three nontrivial permutation $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -modules which are not free, and F for the free $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -module of rank 1, the multiplicities of the various modules in the symmetric algebra can be given in terms of Poincaré series:

Module	Poincaré Series
k	$(1 - t^4)^{-2}(1 + t^3)$
$\Omega^{-2}k$	$(1 - t^4)^{-2}t$
Ω^2k	$(1 - t^4)^{-2}t^2$
P	$(1 - t^4)^{-2}(1 - t^2)(t^2 + t^3 + t^4 + t^5)$
F	$(1 - t^4)^{-2}(1 - t)^{-2}[(1 - t)^{-1}t^2 + 4t^3 + 4t^5 + (1 - t)^{-1}3t^6]$

It turns out that, up to projective summands, $S^{2j+1}\Omega^{-2}k \approx \Omega^{-2}S^{2j}\Omega^{-2}k$, and that the differential $d_2^{i,2j+1}$ is the natural isomorphism $H^i(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, S^{2j+1}) \approx H^{i+2}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, S^{2j})$. (To study this differential in an ad hoc manner one can use the restrictions to index 2 subgroups of $V(2)$ —all of these are isomorphic to $((\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes \mathbb{Z}/4\mathbb{Z}$, and these groups are well understood since they are not far from being wreath products; in fact the cohomology of these groups is detected on abelian subgroups.) As in the previous section, the spectral sequence associated to the extension collapses at E_3 and $E_3^{p,q} = 0$ if $p > 1$.

Remark 6.3. The interest of the method above lies in the odd coincidences that arise in the computations—that $\Omega^{-2}S^{2j}$ should be isomorphic to S^{2j+1} , and that the differential d_2 should contain the associated cohomology isomorphism. A similar phenomenon for the group $X(2)$ was observed in the previous section. It is natural to wonder if these facts can be obtained by some simple method that does not involve ad hoc computations.

In any case, whether from a computer analysis, or following the outline above, one can obtain a number of facts about the cohomology of $V(2)$, which are catalogued below.

Proposition 6.4. *The Poincaré series of $H^*(V(2))$ is*

$$\frac{1 - x + x^2}{(1 - x)^3(1 - x^2)^2},$$

which expands to $1 + 2x + 6x^2 + 11x^3 + 22x^4 + 36x^5 + 60x^6 + 90x^7 + 135x^8 + 190x^9 + 266x^{10} + \dots$

Proposition 6.5. *$H^*(V(2))$ is detected on abelian subgroups.*

Remark 6.6. Note that the proposition above does *not* claim detection on elementary abelian subgroups.

We end our analysis of the groups $V(n)$ on a topological note. One of the key properties of finite 2-groups with central involutions is that they act freely on a product of s spheres, where

s is the rank of the largest elementary abelian subgroup. This number is known to be minimal for this property and interesting results can be derived from this (see [4]). One can prove that the universal V -groups also satisfy this, whence they seem to be rather special. In fact one can use the extension $1 \rightarrow U_r \rightarrow V(n) \rightarrow W(n) \rightarrow 1$ to prove:

Proposition 6.7. *Let $s = \dim \Omega^2(\mathbb{F}_2)$ denote the rank of the largest elementary abelian subgroup in $V(n)$. Then there exist 2^n -dimensional real representations Z_1, \dots, Z_s of $V(n)$ such that its diagonal action on the product of associated linear spheres $X = \mathbb{S}(Z_1) \times \dots \times \mathbb{S}(Z_s)$ is free.*

Proof. Let x_1, \dots, x_r denote a basis for the subgroup U_r . For each i one can construct a 1-dimensional real representation of the Frattini subgroup of $V(n)$ which restricts non-trivially to $\langle x_i \rangle$. This can then be induced to yield a 2^n -dimensional representation Z_i for $V(n)$. Doing the analogous construction for $W(n)$ and pulling back to $V(n)$, we obtain the representations Z_{r+1}, \dots, Z_s . It is elementary to verify that every involution in $V(n)$ must act freely on the product of associated spheres; indeed the unique maximal elementary abelian subgroup satisfies this by construction. \square

7. APPENDIX : AN ELEMENTARY PROOF OF HALF OF THEOREM 4.1

In this section we will provide an elementary proof of “half” of Theorem 4.1 (i.e. the proposition below), and in doing so develop a connection between J_2/J_1 and certain triple commutators.

Proposition 7.1. *If $\{[j_{i_k}] \mid 1 \leq k \leq l\}$ is a finite set of elements in J_1 , and there is an element $[j] \in J$ such that*

1. $[\sigma_{i_k}(j)/j] = [j_{i_k}]$ for each $k \in \{1, \dots, l\}$ and
2. $[\sigma_u(j)] = [j]$ for each $u \notin \{i_1, \dots, i_l\}$,

then

$$\sum_{k=0}^l [a_{i_k}] d_2^{0,1}([j_{i_k}]) = 0.$$

Remark 7.2. The observant reader will note that $d_2^{0,1}$ is merely a notation for a homomorphism determined by our W -group, and that we make no real use of spectral sequences.

Since this is an alternate proof, for the sake of simplicity we will assume $|\dot{F}/\dot{F}^2| = 2^n$; this means that we can simplify the notation by assuming that $\{i_1, \dots, i_l\} = \{1, \dots, n\}$. We leave to the reader the minor modifications necessary for the case in which \dot{F}/\dot{F}^2 is infinite. Recall from the notational conventions in the proof of Theorem 4.1 that to each $[j] \in J_2/J_1$ we can associate an element $\lambda_j \in H^1(G_F^{[2]}) \otimes J_1$, which we shall assume to have the form: $\lambda_j = \sum_{k=1}^n [a_k] \otimes [j_k]$. With the notation of this paragraph, equations 1 and 2 of the proposition above become $[\sigma_k(j)/j] = [j_k] \in J_1$ for each $k = 1, 2, \dots, n$.

Notation 7.3. $F_j := F^{(3)}(\sqrt{j})$, which is a Galois extension of F , and $G_j := \text{Gal}(F_j/F)$.

We will identify the elements of J_1 with “ k -invariants of \mathcal{G}_F ” via $d_2^{0,1}$.

Notation 7.4. In the situation above, if in the expression of $[j] \in J_1$ as a sum of monomials, the coefficient of $[a_i][a_j]$ is nonzero, we will say: “ $[a_i][a_j]$ enters the expression of $[j] \in J_1$ ”, or more concisely “ $[a_i][a_j]$ enters $[j]$.”

We will now use the Kummer pairing $J_1 \times \Phi \rightarrow \mu_2$, given by $\langle [j], \gamma \rangle = \gamma(\sqrt{j})/\sqrt{j}$. From the discussion of this pairing in [24, pp. 42–48] we have:

Lemma 7.5. *Let $1 \leq i \leq k \leq n$, then*

1. $\langle [j], [\sigma_i, \sigma_k] \rangle = -1$ iff $[a_i][a_k]$ enters $[j]$.
2. $\langle [j], \sigma_i^2 \rangle = -1$ iff $[a_i][a_i]$ enters $[j]$.

Notation 7.6. We denote by θ the nontrivial element of the kernel of quotient map $G_j \twoheadrightarrow \mathcal{G}_F$.

Let $\hat{\sigma}_k$, $k = 1, \dots, n$ be extensions of the generators σ_k of \mathcal{G}_F to G_j . Let $\{\hat{\sigma}_i, \hat{\sigma}_k, \hat{\sigma}_l\}$, $1 \leq i < k < l \leq n$, be a triple consisting of generators of G_j . A direct calculation proves:

Lemma 7.7. *Using the terminology of 7.4, we have the following identities:*

1. $[\hat{\sigma}_i, [\hat{\sigma}_k, \hat{\sigma}_l]] = \theta$ iff $[a_k][a_l]$ enters $[j_i]$.
2. $[\hat{\sigma}_i, \hat{\sigma}_k^2] = \theta$ iff $[a_k][a_k]$ enters $[j_i]$.
3. $[a_i][a_i]$ does not enter $[j_i]$.

Another combinatorial lemma which we will need is:

Lemma 7.8. *For σ, τ, γ elements of G_j , we have the following additional identities:*

1. $[\sigma, [\tau, \gamma]][\tau, [\gamma, \sigma]][\gamma, [\sigma, \tau]] = 1$.
2. $[\sigma, \tau]^2 = 1$ and $[\sigma, \tau] = [\tau, \sigma]$
3. $[\sigma^2, \tau] = [\sigma, [\sigma, \tau]] = [\tau, \sigma^2]$

Observe that part 1. of this lemma is the Jacobi identity, which is valid in any metabelian group (see [6]).

Proof of Proposition 7.1. Suppose that $[j] \in J$ is as in the statement of our Proposition. In order to show that $\sum_{\beta=1}^n [a_\beta] d_2^{0,1}([j_\beta]) = 0$ it is enough to observe that for each monomial $[a_i][a_k][a_l]$ that can occur in the expression of the left hand side as an element of $H^3(G_F^{[2]})$, the corresponding coefficient is zero. This can be proved using a case by case analysis, in which division into cases depends on the multiplicities of the $[a_i]$ which appear in the monomial. We limit ourselves to providing a complete analysis for the case where all the multiplicities are 1; the other cases are very similar and left to the reader.

Assume that $[1 \leq i < k < l \leq n]$ then the term $[a_i][a_k][a_l] \in H^3(G_F^{[2]})$ can occur as the summand of the following terms of our sum $\sum_{\beta=1}^n [a_\beta] d_2^{0,1}([j_\beta])$:

$$[a_i] d_1^{0,1}[j_i], \quad [a_k] d_2^{0,1}[j_k] \quad \text{and} \quad [a_l] d_2^{0,1}[j_l].$$

From Lemma 7.7.1 we see that when this term does occur, one of our triple commutators $[\hat{\sigma}_i, [\hat{\sigma}_k, \hat{\sigma}_l], [\hat{\sigma}_k, [\hat{\sigma}_i, \hat{\sigma}_l]]$, or $[\hat{\sigma}_l, [\hat{\sigma}_i, \hat{\sigma}_k]]$ will be θ and not the identity. However from the identity 7.8.1:

$$[\hat{\sigma}_k, [\hat{\sigma}_i, \hat{\sigma}_l]][\hat{\sigma}_i, [\hat{\sigma}_k, \hat{\sigma}_l]][\hat{\sigma}_l, [\hat{\sigma}_i, \hat{\sigma}_k]] = 1$$

and the relation $\theta^2 = 1$ we see that the term $[a_i][a_k][a_l]$ occurs in our sum $\sum_{\beta=1}^n [a_\beta]d_2^{0,1}[j_\beta]$ either zero or two times.

This completes the analysis of the case when all the multiplicities are 1; an analysis of the other cases proves that all terms $[a_i][a_k][a_l]$ occur an even number of times in the sum and thus we have

$$\sum_{\beta=1}^n [a_\beta]d_2^{0,1}([j_\beta]) = 0$$

as desired. □

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MATHEMATICS DEPARTMENT, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN, 53706
E-mail address: adem@math.wisc.edu

4426 S. 112TH E. AVE., TULSA, OK
E-mail address: wgao@sprintmail.com

MATHEMATICS DEPARTMENT, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN, 53706
E-mail address: dikran@math.wisc.edu

MATHEMATICS DEPARTMENT, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA N6A 5B7
E-mail address: minac@math.uwo.ca.edu